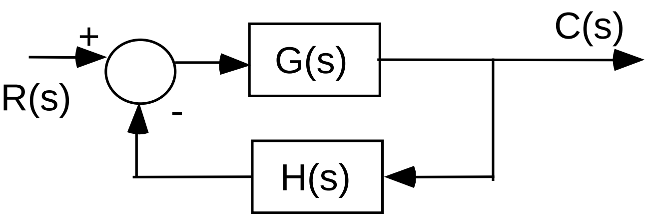
# 23. Root Locus (revisited)

## 23.1 Closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$




Stability is determined by the characteristic equation

$$1+G(s)\,H(s) = 0$$

- Direct computation of roots is tedious.
- Routh's criterion can be used for determining if the system is stable or unstable. But this does not indicate the degree of stability (e.g. amount of overshoot, settling).

  There are two basic satisfactory methods:

  (i) Nyquist plot for steady-state sinusoidal response

  (ii) Root locus = a plot of the roots of the characteristic equation of the closed-loop system as a function of the gain.

    1. Roots of the characteristic equation. can be obtained directly $\Rightarrow$ complete and accurate solution of the transient and steady-state response.

    2. An approximate solution may be obtained with a reduction of the work required.

## 23.2 Basic Conditions of Root Loci

Let the loop transfer function be

$$G(s)\,H(s) = \frac{K(s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = KG_1(s)H_1(s)$$

where $G_1(s)H_1(s)$ no longer contains the variable parameter K.

Substituting this into 1 + G(s) H(s) = 0, one gets

$$F(s) = s_n + s^{n-1} + \cdots + a_{n-1}s + a_n + K(s^m + b_1 s^{m-1} + \cdots + b_m) = 0$$

**Root Loci**: $0 \le K < \infty$

**Complementary loci**: $-\infty < K \le 0$

Complete root loci: root loci & complementary root loci are based on the following condition:

$$1 + KG_1(s)H_1(s) = 0 \quad \Rightarrow \quad G_1(s)H_1(s) = -\frac{1}{K}$$

The following conditions must be satisfied:

$$G_1(s)H_1(s) = \frac{1}{|K|} \qquad -\infty < K < \infty$$

$$\angle G_1(s)H_1(s) = (2\ell + 1)\pi \qquad K \ge 0$$
$$\angle G_1(s)H_1(s) = 2\ell\pi \qquad K < 0$$

where $\ell$ = 0, +1, +2, +3 …

## An Example

$$G(s)\,H(s) = KG_1(s)H_1(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$\left|G_1(s)H_1(s)\right| = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)} = \frac{1}{|K|} \qquad -\infty < K < \infty$$

$$G_1(s)H_1(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{i=1}^{n}\angle(s+p_i) = (2\ell+1)\pi; \quad 0 \le K < \infty$$

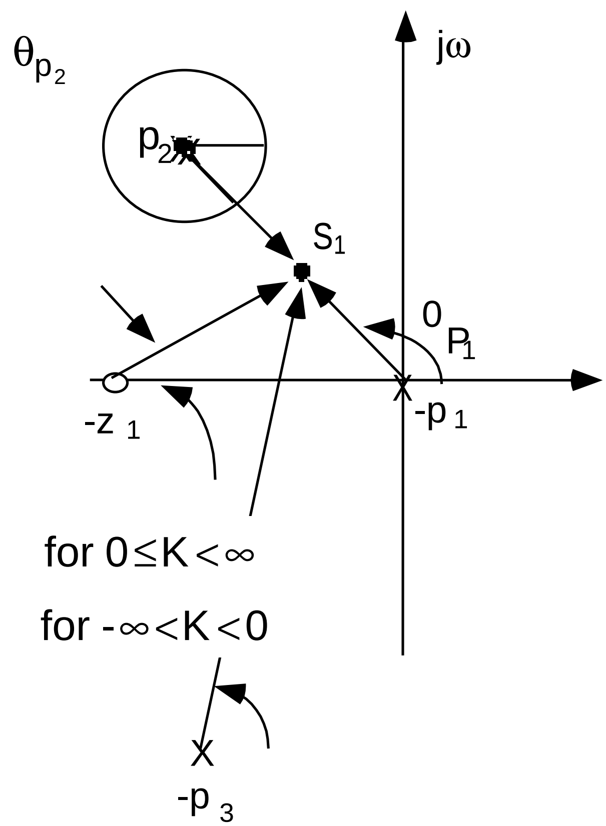
or $\qquad = 2\ell\,\pi; \qquad -\infty < K \le 0$

## Another Example

$$G(S)H(s) = \frac{K(s+z_1)}{s(s+p_2)(s+p_3)}$$

Let $S_1$ be a point on the root loci $(0 \le K < \infty)$

$$\frac{|s+z_1|}{|S_1||S_1+p_2||S_1+p_3|} = \frac{1}{|K|}$$




$$\theta_{z_1} - (\theta_{p_1} + \theta_{p_2} + \theta_{p_3}) = (2\ell+1)\pi \quad \text{for } 0 \le K < \infty$$

$$= 2\ell\pi \quad \text{for } -\infty < K < 0$$

## 23.3 Construction of the Complete Root Loci

We have developed from the relationship between the poles and zeros of G(s) H(s) and the zeros of 1+H G(s) H(s)

**23.3.1** K=0: Points on the complete root loci are at the poles of G(s)H(s).

proof

$$\left| G_1(s)H_1(s) \right| = \frac{\prod_{i=1}^{m} s + z_i}{\prod_{j=1}^{n} s + p_j} = \frac{1}{|K|} \qquad (*)$$

As K $\to$0 (*)$\to\infty$

$\Rightarrow$ s $\to$ poles of $G_1(s)H_1(s)$ or of G(s)H(s)

**23.3.2** K= ± ∞ Points on the complete loci are at the zeros of G(S) H(s)

**23.3.3** The number of branches of the root loci of the system = max(n,m)

e.g. s(s+2) (s+3) + K(s+1) = 0 $\Rightarrow$ 3 root loci

Note: a branch = the locus of one root when K takes on values between - ∞ and ∞ or sometimes between 0 and ∞.

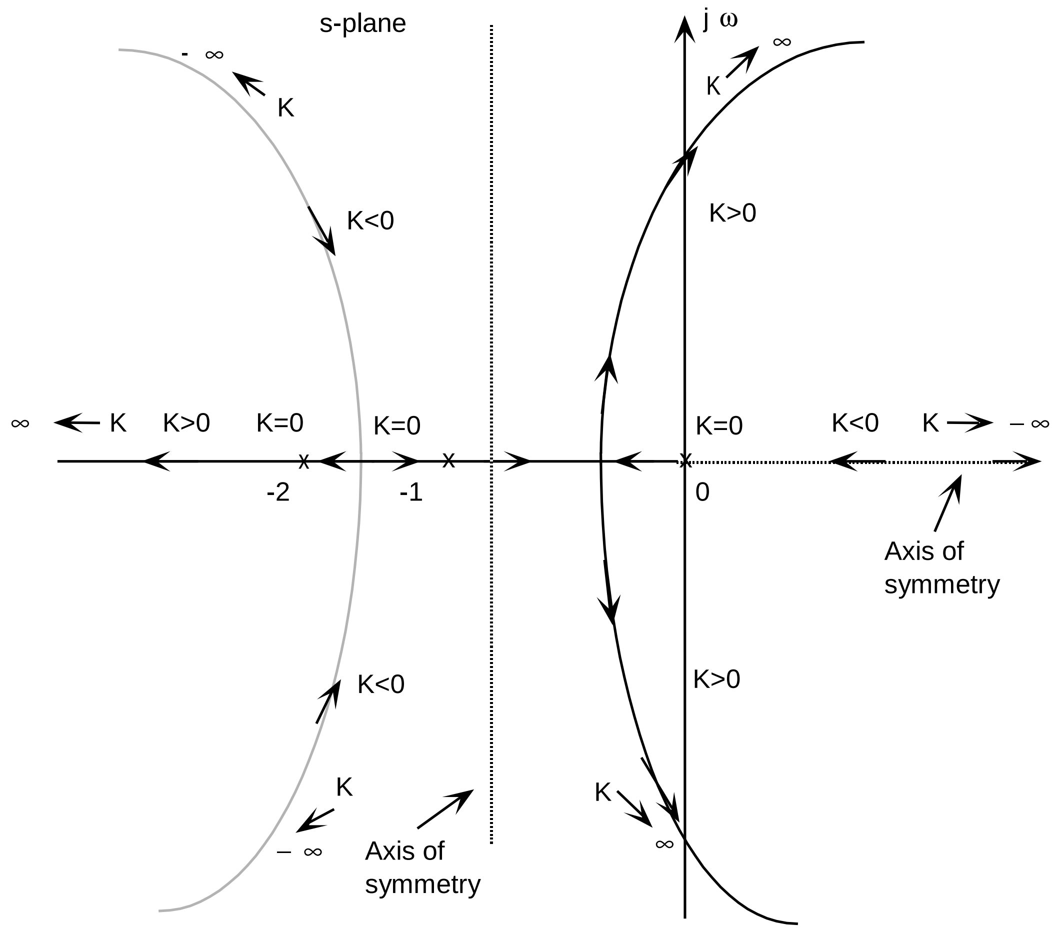
$\Rightarrow$ # of branches = # of roots of 1+ G(s)H(s) = 0

**23.3.4** The <u>complete loci</u> are symmetrical w.r.t. the real axis of the s-plane. In general, the complete root loci are symmetrical with respect to the <u>axes of symmetry</u> of the poles and zeros of G(s) H(s)

**Example**

$$s(s+1)(s+2) + K = 0$$

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$




Root Loci of s(s+1)(s+2)+K=0, showing the properties of symmetry

<u>Remark:</u> When the pole-zero configuration of G(s) H(s) is symmetrical with respect to a point, the complete root loci will also be symmetrical to that point.

<u>ex.</u> $G(s)H(s) = K/s(s+2)(s+1+j)(s+1-j)$

**23.3.5** Asymptotes of the complete root loci (behavior of the loci at $s=\infty$) When $n \neq m$, $2|n-m|$ of the loci will approach $\infty$ in the S-plane.

- For large values of s, the root loci for $K \geq 0$ are asymptotic to straight lines or asymptotes with angles given by

$$Q_\ell = \frac{(2\ell+1)\pi}{n-m}$$

where $\ell = 0,1,2,\cdots,|n-m|-1$

- For the complementary root loci, $K < 0$, the angles of the asymptotes are

$$Q_\ell = \frac{2\pi}{n-m}$$

where $\ell = 0,1,2,\cdots,|n-m|-1$

- Intersection of the asymptotes
  - (a) The intersection of the $2|n-m|$ asymptotes of the complete root loci lies on the real axis of the s- plane.
  - (b) The intersection of the asymptotes is given by

$$\sigma_1 = \frac{b_1 - a_1}{n-m}$$

$$\frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m}$$

**23.3.6** Root Loci on the Real Axis

(i) On a given section of the real axis, root loci (K>0) may be found in the section only if the total number of real poles and zeros of G(s) H(s) to the right of the section is odd.

(ii) On a given section of the real axis, complementary root loci (k<0) may be found in the section only if the total # of real poles and zeros of G(s) H(s) to the right of the section is even.

**23.3.7** Angles of departure (from poles) and the angles of arrival at zeros of the complete root loci. See previous notes.

**23.3.8** Breakaway Points (saddle points) on the Complete Root Loci.


Breakaway points must either be real or occur in complex-conjugate pairs.The breakaway points on the complete root loci of $1 + KG_1(s)H_1(s) = 0$ must satisfy

$$\frac{dG_1(s)H_1(s)}{ds} = 0$$

## 23.4 Calculation of K on the Root Loci

At a point $s = s_1$ on the loci

$$|K| = \frac{1}{|G_1(s_1)H_1(s_1)|} = \frac{\prod_{j=1}^{n} s_1 + p_j}{\prod_{i=1}^{m} s_1 + z_j}$$